Useful Notes on Mathematics

V. G. Kohn
Russian Research Centre "Kurchatov Institute", 123182 Moscow.

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Abstract

This article contains useful mathematic relations from mathematical reference books as well as my own inventions in this fields. It is assumed that this file will grow to be a useful reference article.

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- Fast Fourier Transformation
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Fast Fourier Transformation

The FFT procedure is used for integrals in infinite limits

$$\widetilde{F}(q) = \int_{-\infty}^{\infty} dx \exp(-iqx) F(x)$$

if the function has zero tails in the regions of large argument module. In this case the limits can be replaced by finite value and the intergal has a form

$$\widetilde{F}(q) = \int_{-X/2}^{X/2} dx \exp(-iqx) F(x)$$

The region of integration X is divided on $N = 2^k$ subregions which central points has a form

$$x_k = d * (k - (N-1)/2), \quad d = X/N, \quad k = 0, ..., N-1$$

In addition let us consider a special set of points in q-space

$$q_m = p * (m - (N-1)/2), \quad p = Q/N, \quad m = 0, ..., N-1$$

and apply a special condition

$$dp = \frac{2\pi}{N}, \quad p = \frac{2\pi}{X}, \quad Q = \frac{2\pi N}{X} = \frac{2\pi}{d}$$

For such set of points we have

$$-q_m x_k = -\frac{2\pi km}{N} + \pi(1 - \frac{1}{N})(k+m) - \pi\frac{N}{2} + \pi - \frac{\pi}{2N}$$

Therefore the integral can be replaced by the sum

$$\widetilde{F}_m = AdC_m \sum_{k=0}^{N-1} \exp\left(-i\frac{2\pi km}{N}\right) F_k C_k, \quad C_k = \exp\left(i\pi k - i\frac{\pi k}{N}\right), \quad A = \exp\left(-i\frac{\pi N}{2} + i\pi - i\frac{\pi}{2N}\right)$$

Similar expression can be obtained for back Fourier Transformation

$$F(x) = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \exp(iqx)\widetilde{F}(q)$$

and it looks as

$$F_k = A^* \frac{p}{2\pi} C_k^* \sum_{m=0}^{N-1} \exp\left(i\frac{2\pi km}{N}\right) \widetilde{F}_m C_m^*,$$

The calculation of a convolution of some function with the propagator having analytical Fourier image $\widetilde{P}(q)$ can be performed by double FFT with inserting $\widetilde{P}_m$ inside a second sum. In this way we see that the multipliers A and A^* can be omitted because $|A| = 1$. Multiplications by C_m and then by C_m^* can be omitted as well for the same reason.

Kirchhoff propagator from the plane surface

Let us consider a solution of Maxwell's equation in an empty half-space $z > 0$ when the boundary condition is installed on the plane (x, y). Let the known wave field at the plane has a structure $\exp(iKSx)F(x, y)$ where $K = 2\pi/\lambda$ is a wave number, $S = \sin\varphi$ where φ is an angle between the normal to the surface (z-axis) and some direction in space (direction of wave movement) and $F(x, y)$ is a slow function as compared to the exponential. The standard solution of the task based on the Fourier decomposing the function $F(x, y)$. So we may represent the boundary condition at $z = 0$ as

$$\exp(iKSx)F(x, y) = \int \frac{dq_x}{2\pi} \int \frac{dq_y}{2\pi} \exp(iKSx + iq_x x + iq_y y) f(q_x, q_y) \tag{1}$$

where $f(q_x, q_y)$ is a Fourier image of $F(x, y)$.

When $z > 0$ the solution $G(x, y, z)$ may be written in the same form of double Fourier integral with an additional exponential factor depending on z, namely,

$$\begin{aligned} G(x, y, z) &= \int \frac{dq_x}{2\pi} \int \frac{dq_y}{2\pi} \exp(iKSx + iq_x x + iq_y y) f(q_x, q_y) \times \\ &\quad \times \exp\left(iz\sqrt{K^2 - (q_x + KS)^2 - q_y^2}\right) \end{aligned} \tag{2}$$

Indeed, it is easy to obtain that the exponential function exp() inside the integral satisfied the equation $[\nabla^2 + K^2]\exp() = 0$ for each value of q_x and q_y. Since the equation is linear the integral is also a solution of the equation.

We assume that $F(x, y)$ is a slow function. Therefore $f(q_x, q_y)$ is significant only for small values of q_x and q_y as compared to K and we may use an expansion of square root up to second degrees of q_x and q_y

$$\sqrt{K^2 - (q_x + KS)^2 - q_y^2} \approx K\gamma - q_x\frac{S}{\gamma} - \frac{1}{2K}\left(\frac{q_x^2}{\gamma^3} + \frac{q_y^2}{\gamma}\right) \tag{3}$$

where $\gamma = \cos\varphi = \sqrt{1 - S^2}$. Now the solution may be written as follows

$$G(x, y, z) = \int \frac{dq_x}{2\pi}\int \frac{dq_y}{2\pi} f(q_x, q_y)\exp(i\Phi[q_x, q_y, x, y, z) \tag{4}$$

with

$$\Phi = K(Sx + \gamma z) + q_x(x - z\frac{S}{\gamma}) + q_y y - \frac{z}{2K\gamma}\left(\frac{q_x^2}{\gamma^2} + q_y^2\right)$$

A structure of the phase allows one to make a conclusion that the point $x = 0$, $y = 0$ at the boundary plane ($z = 0$) will move to the point $x = zS/\gamma$, $y = 0$ when $z > 0$. A locus of such points makes a line that goes along the vector $\mathbf{k}$ with the components $k_x = KS$, $k_y = 0$, $k_z = K\gamma$. Let us choose the new origin at the point $x_0 = z_0 S/\gamma$, $y_0 = 0$, $z = z_0$ at this line and consider a new coordinat system x_d, y_d and z_d so that the axis z_d does along the wave vector $\mathbf{k}$. The transformation looks as follows

$$x - x_0 = x_d\gamma + z_d S, \quad y = y_d, \quad z - z_0 = -x_d S + z_d\gamma \tag{5}$$

The phase Φ now may be written as

$$\Phi = KL + Kz_d + \frac{q_x}{\gamma}x_d - \frac{L}{2K}\left(\frac{q_x}{\gamma}\right)^2 + q_y y_d - \frac{L}{2K}q_y^2 \tag{6}$$

where $L = z_0/\gamma$ is the distance between the old and new coordinate systems and we neglect small coordinates x_d, y_d and z_d in terms with q_x^2,.q_y^2 as compared to the long distance L. The solution in the plane $z_d = 0$ of new coordinate system is

$$\begin{aligned} G(x_d, y_d) &= \exp(iKL)\gamma\int \frac{dq}{2\pi}\exp(iqx_d)\exp\left(-i\frac{L}{2K}q^2\right)\times \\ &\quad \times\int \frac{dq_y}{2\pi}\exp(iq_y y_d)\exp\left(-i\frac{L}{2K}q_y^2\right) f(q\gamma, q_y) \end{aligned} \tag{7}$$

where $q = q_x/\gamma$.

The formula (7) may be transformed into the real space taking into account that

$$\begin{aligned} \exp\left(-i\frac{L}{2K}q^2\right) &= \int dx\exp(-iqx)P(x, L), \\ f(q\gamma, q_y) &= \int dxdy\exp(-iqx\gamma - iq_y y)F(x, y) \end{aligned} \tag{8}$$

where

$$P(x,z) = \frac{1}{\sqrt{i\lambda z}} \exp\left(i\pi \frac{x^2}{\lambda z}\right), \quad \lambda = \frac{2\pi}{K}, \tag{9}$$

is a partial propagator of the transverse wave field profile of one dimension in a small angle approxiamtion. Finally we have a relation

$$G(x_d, y_d) = \exp(iKL)\gamma \int dxdy P(x_d - x\gamma, L) P(y_d - y, L) F(x,y) \tag{10}$$

It is a result of this section and a solution of the task. We note that fast exponential factor $\exp(iKSx)$ in the boundary conditions leads to a choice of the optical axis along the vector $\mathbf{k}$. It disappears in the plane normal to the vector $\mathbf{k}$. On the other hand, since the vector $\mathbf{k}$ makes an angle φ with the normal to the boundary, the parameter $\gamma = \cos\varphi$ appears before the integral and in the argument of the propagator.

Integrals of spherical wave optics

A general form of the integral of a parabolic phase exponent

$$\int_a^b dx\, \exp(i\alpha - i\beta x + i\gamma x^2) = \frac{\exp(i\alpha - i\gamma B^2)}{A} \left[\, F(A[b+B]) - F(A[a+B]) \,\right], \tag{11}$$

where

$$A = \sqrt{\frac{2\gamma}{\pi}}\,, \qquad B = \frac{\beta}{2\gamma}\,.$$

Here $F(x)$ is the complex Fresnel integral

$$F(x) = \int_0^x dt\, \exp(i\,\frac{\pi}{2}\,t^2)\,, \qquad F(-x) = -F(x)\,. \tag{12}$$

The asimptotic series of this function has the form

$$\begin{aligned} F(z) &= \left(\frac{i}{2}\right)^{\frac{1}{2}} + \frac{\exp(i\pi z^2/2)}{i\pi z} \left[1 + \frac{1}{i\pi z^2} + \frac{1\cdot 3}{(i\pi z^2)^2} + \frac{1\cdot 3\cdot 5}{(i\pi z^2)^3} + \cdots\right], \\ |z| &\to \infty\,, \qquad \arg(z) < \pi/2\,. \end{aligned} \tag{13}$$

Taking this into account one obtains easily the particular cases

$$\int_{-\infty}^{\infty} dx\, \exp(i\alpha - i\beta x + i\gamma x^2) = \left(\frac{i\pi}{\gamma}\right)^{1/2} \exp\left(i\alpha - i\frac{\beta^2}{4\gamma}\right) \tag{14}$$

and

$$\begin{aligned} \int_0^{\infty} dx\, \exp(i\alpha - i\beta x + i\gamma x^2) &= \frac{1}{2}\left(\frac{i\pi}{\gamma}\right)^{1/2} \exp\left(i\alpha - i\frac{\beta^2}{4\gamma}\right) \times \\ &\quad \times \left[1 - \left(\frac{2}{i}\right)^{1/2} F\left(\frac{\beta}{\sqrt{2\pi\gamma}}\right)\right] \end{aligned} \tag{15}$$

Normalized Gauss function.

The expression looks as follows

$$G(x) = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

The value FWHM w can be estimated as

$$w = \sigma(8\log 2)^{1/2} = 2.3548\sigma$$

Bessel functions.

There is a problem to calculate the Bessel functions for intermediate values of the argument from a power series and a asimptotic series. The useful approach is described below. Let us consider a standard recurrent formula for a Bessel functions

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)\,. \tag{16}$$

It is convenient to introduce the new help function

$$r_n(x) = \frac{J_n(x)}{J_{n-1}(x)} \tag{17}$$

which allows to represent the function of higher order through $J_0(x)$

$$J_n = r_n \cdots r_2\, r_1\, J_0\,. \tag{18}$$

A substitution 18 in 16 gives

$$r_n = \frac{x}{2n - x\, r_{n+1}}\,. \tag{19}$$

This form of the reccurent relation is convenient because if one starts from a very large index N and use this relation N times then one obtains all values r_n with a good accuracy independently on r_N. This is owing to the fact that for large n one has $x\, r_n(x) \ll 2n$.

On the other hand a next series takes place for the Bessel functions

$$1 = J_0 + 2\, J_2 + 2\, J_4 + 2\, J_6 + \cdots \tag{20}$$

which can be rewritten in terms of r_n as

$$J_0 = \frac{1}{1 + 2r_1r_2(1 + r_3r_4(1 + \cdots))}\,. \tag{21}$$

From 18 and 21 we have finally

$$J_n = \frac{r_1r_2\cdots r_n}{1 + 2r_1r_2(1 + r_3r_4(1 + \cdots))} \tag{22}$$

as a good tool for a computer calculation of the Bessel function. The calculating time depends on the value of N which depends on the index n and argument x.

The functions $J_0(x)$ and $J_1(x)$ has a very good polynomial approximation for small and large argument regions separately.

$$|\,x\,| < 3\,, \qquad z = (x/3)^2$$

$$\begin{aligned} J_0(x) &= 1 - z(a_1 - z(a_2 - z(a_3 - z(a_4 - z(a_5 - a_6 z))))) + \varepsilon\,, \\ \varepsilon &< 5 \cdot 10^{-8}\,, \end{aligned} \tag{23}$$

where
$a_1 = 2.2499997\,, \quad a_2 = 1.2656208\,, \quad a_3 = 0.3163866\,,$
$a_4 = 0.0444479\,, \quad a_5 = 0.0039444\,, \quad a_6 = 0.0002100\,.$

$$\begin{aligned} x^{-1}J_1(x) &= 1/2 - z(a_1 - z(a_2 - z(a_3 - z(a_4 - z(a_5 - a_6 z))))) + \varepsilon\,, \\ \varepsilon &< 1.3 \cdot 10^{-8}\,, \end{aligned} \tag{24}$$

where
$a_1 = 0.56249985\,, \quad a_2 = 0.21093573\,, \quad a_3 = 0.03954289\,,$
$a_4 = 0.00443319\,, \quad a_5 = 0.00031761\,, \quad a_6 = 0.00001109\,.$

$$3 < |\,x\,| < \infty\,, \quad z = 3/x$$

$$\begin{aligned} J_0(x) &= x^{-1/2} f_0(x)\,\cos\theta_0(x)\,, \\ f_0(x) &= 0.79788456 - z(a_1 + z(a_2 + z(a_3 - z(a_4 - z(a_5 - a_6 z))))) + \varepsilon\,, \\ \varepsilon &< 1.6 \cdot 10^{-8}\,, \end{aligned} \tag{25}$$

where
$a_1 = 0.00000077\,, \quad a_2 = 0.00552740\,, \quad a_3 = 0.00009512\,,$
$a_4 = 0.00137237\,, \quad a_5 = 0.00072805\,, \quad a_6 = 0.00014476\,,$

$$\begin{aligned} \theta_0(x) &= x - 0.78539816 - z(a_1 + z(a_2 - z(a_3 - z(a_4 + z(a_5 - a_6 z))))) + \varepsilon\,, \\ \varepsilon &< 7 \cdot 10^{-8}\,, \end{aligned}$$

where
$a_1 = 0.04166397\,, \quad a_2 = 0.00003954\,, \quad a_3 = 0.00262573\,,$
$a_4 = 0.00054125\,, \quad a_5 = 0.00029333\,, \quad a_6 = 0.00013558\,.$

$$\begin{aligned} J_1(x) &= x^{-1/2} f_1(x)\,\cos\theta_1(x)\,, \\ f_1(x) &= 0.79788456 + z(a_1 + z(a_2 + z(a_3 - z(a_4 - z(a_5 - a_6 z))))) + \varepsilon\,, \\ \varepsilon &< 4 \cdot 10^{-8}\,, \end{aligned} \tag{26}$$

where
$a_1 = 0.00000156\,, \quad a_2 = 0.01659667\,, \quad a_3 = 0.00017105\,,$
$a_4 = 0.00249511\,, \quad a_5 = 0.00113653\,, \quad a_6 = 0.00020033\,.$

$$\begin{aligned} \theta_1(x) &= x - 2.35619449 + z(a_1 + z(a_2 - z(a_3 - z(a_4 + z(a_5 - a_6 z))))) + \varepsilon\,, \\ \varepsilon &< 9 \cdot 10^{-8}\,, \end{aligned}$$

where
$a_1 = 0.12499612\,,\quad a_2 = 0.00005650\,,\quad a_3 = 0.00637879\,,$
$a_4 = 0.00074348\,,\quad a_5 = 0.00079824\,,\quad a_6 = 0.00029166\,.$

On the problem of numerical integration.

Let us consider the case when one has the array of values of any function $f(x_n) = f_n$ on the set of nodes with a constant step d, namely, $x_{n+1} = x_n + d$. The problem is to calculate the integral of this function from one node to any another. If the interval (a, b) consists of n steps d, namely, $(b - a) = nd$, then one can use the Newton-Kotes formula with $N = n+1$ nodes which is based on a search of the polynomial having the same values f_n at these nodes. The analytical integration of this polynomial gives the approximate result for the integral. It is known that the coefficients of this polynomial will have large values with different signs and this way is inconvenient for a numerical calculation.

Another way is the well known multiple use of the trapezium rule, simpson rule or 3/8 rule when the interval is divided on m similar subintervals containing 2,3 or 4 nodes. These formulas have the next form:
Trapezium rule (N is any, $x_1 = a$, $x_N = b$)

$$\int_{x_1}^{x_N} dx\, f(x) = d\left[\sum_{n=2}^{N-1} f_n + \frac{1}{2}\,(f_1 + f_N)\right] \tag{27}$$

simpson rule ($N = 2m + 1$ is odd)

$$\int_{x_1}^{x_N} dx\, f(x) = \frac{d}{3}\left[4\sum_{n=2,4\ldots}^{N-1} f_n + 2\sum_{n=3,5\ldots}^{N-2} f_n + f_1 + f_N\right] \tag{28}$$

The 3/8 rule ($N = 3m + 1$)

$$\int_{x_1}^{x_N} dx\, f(x) = \frac{3d}{8}\left[3\sum_{n=2,5\ldots}^{N-2} f_n + 3\sum_{n=3,6\ldots}^{N-1} f_n + 2\sum_{n=4,7\ldots}^{N-3} f_n + f_1 + f_N\right] \tag{29}$$

To estimate the accuracy one usually use the new set of nodes with the step which is half of the primary step. By calculating the imtermediate nodes one obtains the new values of the integral and compares it with the old value. It is easy to understand that this estimation is good only for a first value but not for a second one which is more accurate. For this reason it is convenient to use $N = 6m + 1$ from the beginning and to calculate three values of the integral with a use of all these formulas simultaneously. The difference will show the accuracy. If the accuracy turns out to be a not good then one is in a need to calculate the new values of the function at the intermediate nodes.

The goal of this section is to show that the accuracy can be very increased by calculating the only two or four values of the function at the nodes outside of the interval. Indeed, let us consider the integral on the interval from x_n to x_{n+1}. To calculate it let us find the polynomial which coincides with the function in four nodes (n-1, n, n+1, n+2) and then integrate it on (x_n, x_{n+1}). The result has the form

$$\int_{x_n}^{x_{n+1}} dx\, f(x) = d\left[\frac{1}{2}\,(f_n + f_{n+1}) + \frac{1}{24}\,(f_n + f_{n+1} - f_{n-1} - f_{n+2})\right] \tag{30}$$

A multiple use of this formula gives the result which was called as a modified trapezium formula [1]

$$\int_{x_1}^{x_N} dx\, f(x) = d\left[\sum_{n=2}^{N-1} f_n + \frac{1}{2}(f_1 + f_N) + \frac{1}{24}(f_2 + f_{N-1} - f_0 - f_{N+1})\right] \tag{31}$$

Here $f_0,\ f_{N+1}$ means the function value at $x_0 = a-d$, $x_{N+1} = b+d$. This formula gives the accuracy which is comparable with the simpson formula but has a structure of trapezium formula.

To obtain a higher accuracy in this way one can consider the polynomial which equals to the function in six nodes (n-2, n-1, n, n+1, n+2, n+3) for the integration on $(x_n,\ x_{n+1})$. The result has a similar form

$$\int_{x_n}^{x_{n+1}} dx\, f(x) = d\left(\frac{1}{2}(f_n + f_{n+1}) + \frac{1}{24}(f_n + f_{n+1} - f_{n-1} - f_{n+2}) + \right.$$
$$\left. + \frac{1}{1440}\left[22(f_n + f_{n+1}) - 33(f_{n-1} + f_{n+2}) + 11(f_{n-2} + f_{n+3})\right]\right) \tag{32}$$

A multiple use of this formula brings us a new trapezium type formula

$$\int_{x_1}^{x_N} dx\, f(x) = d\left[\sum_{n=2}^{N-1} f_n + \frac{1}{2}(f_1 + f_N) + \frac{41}{720}(f_2 + f_{N-1} - f_0 - f_{N+1}) - \right.$$
$$\left. - \frac{11}{1440}(f_3 + f_{N-2} - f_{-1} - f_{N+2})\right] \tag{33}$$

The comparison 33 with 31 gives the estimation of the accuracy which involves only the function values at the ends of the interval. It is true because the middle nodes represent the function into integral with a very good accuracy having the constant unit weights.

To obtain a more clear understanding of the possibilities of the formulas derived above I performed several numerical calculations of the integrals of different types with a use of all five formulas and fixed number of nodes. Below the formulas 27, 31 and 33 are refered as trapezium-1, trapezium-2 and trapezium-3. The first integral which has been texted has the form

$$\int_0^5 dx\, x^n = \frac{5^{n+1}}{n+1} \tag{34}$$

The results of calculation for different values of n are shown in the table

n	true	N	trap1	trap2	trap3	simp	3/8
0	5.	7	5.	5.	5.	5.	5.
1	12.5	7	12.5	12.5	12.5	12.5	12.5
2	41.667	7	42.245	41.667	41.667	41.667	41.667
3	156.25	7	160.59	156.25	156.25	156.25	156.25
4	625.00	7	653.85	624.12	625.00	625.32	625.72
5	2604.17	7	2784.01	2593.12	2604.17	2608.19	2613.21
6	11161.	7	12236.	11049.	11165.	11200.	11248.
7	48828.	7	55070.	47847.	48895.	49165.	49557.
10	4438921	25	4509417	4437298	4438946	4439506	4440223

This table shows that the best result is obtained by means of new trapezium-3 formula. The second place belongs to simpson formula which gives slightly more good results in comparison with 3/8 formula. The possible reason is the fact that the loss of accuracy at the ends of interval increases with increasing the degree of the polynomial. Indeed, the reference books [1, 2] gives for the simpson rule the better accuracy estimation than for the 3/8 rule despite the 3/8 rule uses the four degree polynomial.

The second integral which has been tested is

$$\int_0^{\pi} dx\, \cos(z \sin x) = \pi\, J_0(z) \tag{35}$$

where $J_0(z)$ is the Bessel function. The results are shown in the table

z	true	N	trap1	trap2	trap3	simp	3/8
2	.703374	13	.703374	.703374	.703374	.703374	.703357
4	-1.24768	13	-1.24768	-1.24768	-1.24768	-1.24770	-1.25085
6	.473266	13	.473266	.473267	.473267	.472125	.428866
10	-.772630	13	-.772629	-.772629	-.772629	-.905351	-1.0235
10	-.772630	25	-.772630	-.772630	-.772630	-.772630	-.773860
20	.524723	25	.524723	.524723	.524723	.482984	.410686

This table shows that the all trapezium type formulas gives much better accuracy for the integral in comparison with the simpson and 3/8 rules. The reason of this fact is that the integrand is the periodical function which has the same values at the ends of the interval and a symmetry about the centre. It is well known that the periodical functions is better represented by the Fourier series than the power series. On the other hand the trapezium rule is accurate for $\sin(nx)$ and $\cos(nx)$ with $N = n + 1$ [2].

Let us consider a more general integrand without a symmetry and with a bad behaviour

$$\int_0^{5} dx\, x\, \cos(zx) = \frac{1}{z^2}\,(5z \sin(5z) + \cos(5z) - 1) \tag{36}$$

The results of calculation are seen as follows

z	true	N	trap1	trap2	trap3	simp	3/8
2	-1.8198	13	-1.7677	-1.8198	-1.8207	-1.8195	-1.8184
4	1.1042	25	1.0352	1.0952	1.1029	1.1079	1.1133
6	-.8468	49	-.8206	-.8451	-.8467	-.8475	-.8484

Once again one can see that trapezium-3 formula is the best one and simpson formula takes a second place.

The last tested integral has the form of local function which equals zero far from the centre

$$\int_{-a}^{a} dx\, \frac{1}{x^2 + z^2} = \frac{\pi}{z} - \frac{2}{a}, \quad a \gg z \tag{37}$$

The results of calculations are shown in the table

a	z	true	N	trap1	trap2	trap3	simp	3/8
50	1	3.1016	193	3.1016	3.1016	3.1016	3.0966	3.0873
50	2	1.5308	97	1.5308	1.5308	1.5308	1.5283	1.5237
25	.5	6.2032	193	6.2033	6.2033	6.2033	6.1932	6.1746

Thus we see once again that the trapezium type formulas are much better than the simpson or 3/8 rules. In this case the ends of interval are not essential because the function equals zero at the ends and the simple trapezium rule can be used. In my opinion there are two explanations of this result. Firstly, the function with zero values at the ends can be treated as periodical function for which the Fourier series is better than power series. Secondly, all integration formulas presented above have a general meaning which do not take into account the specific properties of the integrand. Therefore, it seems strange when some nodes has a weight different from the weight of other nodes for the function having approximately homogeneous behaviour inside the interval and similar behaviour at the ends. It is good only for monotonous function with very different behaviour at the ends. Only for the functions of such type the simpson formula is better than the trapezium formula. However, taking into account four extra nodes outside of the interval makes the trapezium type formula to be better than simpson rule for these functions as well.

The general conclusion of this section can be formulated as the sentence that there are a lot of cases where a simple trapezium rule of integrating is better than the simpsom rule. First of all it is true for the periodical functions and the functions localized in the centre of integration region. The new trapezium-3 formula obtained by addition of four nodes outside the integration region is the best in all cases.

References

[1] Handbook of mathematical functions, 1964, Eds. M.Abramowitz and I.A.Stegun. Applied Mathematic Serie •55

[2] Krylov V. I., Shulgina L. T. Spravochnaya kniga po chislennomu integrirovaniu, M., Nauka, 1966 (in russian)