# On the Theory of X-ray Refractive Optics: Exact Solution for a Parabolic Medium 

V. G. Kohn<br>Russian Research Centre Kurchatov Institute, pl. Kurchatova 1, Moscow, 123182 Russia<br>e-mail: kohn@kurm.polyn.kiae.su<br>Received October 24, 2002


#### Abstract

The exact solution is obtained for a propagator describing x-ray propagation through a refractive parabolic medium. Such a medium arises in compound many-element refractive x-ray lenses that are used in synchrotron radiation sources. The solution obtained allows one to analyze such lenses in detail to predict their operation in particular applications (beam focusing, microobject imaging, and Fourier transform). © 2002 MAIK "Nauka/Interperiodica".


PACS numbers: 41.50.+h; 07.85.Qe; 42.79.Bh; 42.30.-d

For a hundred years after the discovery of x-ray radiation, it was thought that refractive lenses could not be used for focusing hard x-rays because, at least, of two reasons. First, the refractive index for electromagnetic radiation with energy $E$ ranging from 10 to 50 keV differs only slightly from unity. Second, the absorption coefficient for this radiation is nonzero. When writing the complex refractive index in the form $n=1-\delta+i \beta$, one has, e.g., for aluminum at $E=25 \mathrm{keV} \delta=8.643 \times$ $10^{-7}$ and $\beta=1.747 \times 10^{-9}$.

This problem was solved in 1996 [1] with the use of compound lenses, i.e., lenses composed of a large number of relatively thin elements. It proved to be quite fortunate that the x-ray phase velocity in a material is higher than the velocity of light in free space. For this reason, the focusing lens was taken to be biconcave and the thickness of a material in the central part of the lens was smaller than the absorption length. To date, many publications have been devoted to various methods of fabricating compound refractive x-ray lenses. Among them, of primary interest are lenses with circular aperture and parabolic profile. Elements of these prisms are obtained by pressing out parabolic profile in aluminum plates (see, e.g., [2]) or plates of organic materials (see, e.g., [3]). Each element focuses a parallel beam into the point at distance $F_{1}=R / 2 \delta$, where $R$ is the radius of curvature of the parabolic profile (see figure). In this case, the focal length of a block with $N$ elements is $F \approx F_{1} / N$. Let, e.g., $F_{1}=100 \mathrm{~m}$. The focal length of a block with 100 elements will be 1 m , which is quite appropriate for the experiments at synchrotron radiation stations.

A lens containing 1000 or more elements can rather easily be fabricated. The length $L=N p$ of the compound lens increases with the number $N$ of elements, while the focal length $F$ decreases. Clearly, the focal length in the case $L \ll F$ can be estimated from the formula $F \approx R / 2 N \delta$ for a thin lens. However, in this case
the linear corrections in the small parameter $L / F$ may be quite appreciable when imaging microobjects with extreme resolution. The theoretical analysis of the operation of a compound lens with length $L$ comparable to the focal length $F$ was performed only in the geometrical optics approximation (see, e.g., [4]), which is, clearly, insufficient for the estimation of focal spot size and for the analysis of image transfer using this lens.

A complete solution to the problem of radiation transfer through a long compound lens must have the form of an integral relationship of the Kirchhoff integral type. In this case, the problem amounts to determining the kernel of integral transformation (propagator) by solving the Maxwell equation with initial condition in the form of the Dirac delta function. It is shown in this work that, under certain conditions, this problem has an exact solution; i.e., the propagator can be calculated analytically in a form close to the Gaussian function with complex parameters, for which one can write the exact recurrence formulas. It is assumed that the synchrotron radiation (SR) is preliminarily monochromatized and has a rather high degree of spatial coherence. These conditions are fulfilled, e.g., in the thirdgeneration SR sources [5].


[^0]We choose the optical axis along the $z$ axis (figure) and represent the general solution to the Maxwell equation as $E(x, y, z)=\exp (i k z) A_{t}(x, y, z)$, where $k=\omega / c$ is the wave number in vacuum. The function $A_{t}(x, y, z)$ describes the transfer, along the $z$ axis, of the transverse dependence of the wave field. Since the radiation is hard and interacts weakly with a material, one can use, with a high accuracy, the paraxial approximation, i.e., ignore the second derivative of $A_{t}$ with respect to the coordinate $z$, as compared to the first derivative. As a result, one arrives at the parabolic equation for the function $A_{t}(x, y, z)$

$$
\begin{equation*}
\frac{d A_{t}}{d z}=-i k \eta s(x, y, z) A_{t}+\frac{i}{2 k}\left(\frac{d^{2} A_{t}}{d x^{2}}+\frac{d^{2} A_{t}}{d y^{2}}\right) \tag{1}
\end{equation*}
$$

where $\eta=1-n=\delta-i \beta=\delta(1-i \gamma)$. In the radiationtransfer problem, the wave field at the entrance surface of the lens is a given function $A_{t}(x, y, 0)=A_{0}(x, y)$, where the coordinate $z$ is measured from the outset of the lens. In the compound lens, the function $s(x, y, z)$ is unity in the regions inside the material and zero outside it (figure).

In what follows, I consider only the case where the thickness $p$ of an individual element of a compound lens is smaller than the characteristic scale of changing the transverse dependence of the wave field. In other words, the thin-lens approximation is assumed to be fulfilled for an individual element. This is always true for a compound lens with many elements. This restriction can be used for averaging the function $s(x, y, z)$ over its period and replacing it by a function depending only on the transverse coordinates:

$$
\begin{equation*}
\bar{s}(x, y)=\frac{d}{p}+\frac{x^{2}}{p R}+\frac{y^{2}}{p R} \tag{2}
\end{equation*}
$$

This dependence is valid only inside the lens geometrical aperture with diameter $a=2[R(p-d)]^{1 / 2}$ (figure). However, the effective operation area (effective aperture) of the lens is determined by the x-ray absorption in its material and is almost always smaller than the geometrical aperture. Because of this, one can formally assume that dependence (2) holds everywhere over the region of transverse plane $(X, Y)$ considered.

Let us represent the initial wave field as a Fourier integral

$$
\begin{equation*}
A_{0}(x, y)=\int \frac{d q_{x} d q_{y}}{(2 \pi)^{2}} \exp \left(i q_{x} x+i q_{y} y\right) \tilde{A}_{0}\left(q_{x}, q_{y}\right) \tag{3}
\end{equation*}
$$

and consider the solution $\tilde{P}_{t}\left(x, y, q_{x}, q_{y}, z\right)$ with the initial function in the form of plane wave $\tilde{P}_{t}\left(x, y, q_{x}, q_{y}, 0\right)=$ $\exp \left(i q_{x} x+i q_{y} y\right)$. The solution can be represented as the
product $\tilde{P}_{t}=\exp (-i k \eta[d / p] z) \tilde{P}\left(x, q_{x}, z\right) \tilde{P}\left(y, q_{y}, z\right)$, with the partial function $\tilde{P}(x, q, z)$ satisfying the equation

$$
\begin{gather*}
\frac{d \tilde{P}}{d z}=-i \frac{k \eta}{p R} x^{2} \tilde{P}+\frac{i}{2 k} \frac{d^{2} \tilde{P}}{d x^{2}}  \tag{4}\\
\tilde{P}(x, q, 0)=\exp (i q x)
\end{gather*}
$$

This equation coincides formally with the Schrödinger equation for a particle in a parabolic potential. Nevertheless, the expansion in terms of the stationary states will not be considered in this work.

Taking into account the character of the initial function, it is reasonable to seek a solution in the form of a Gaussian function with complex coefficients

$$
\begin{gather*}
\tilde{P}(x, q, z)=\exp \left(i a_{0}(z)+i a_{1}(z) x+i a_{2}(z) x^{2}\right)  \tag{5}\\
a_{0}(0)=a_{2}(0)=0, \quad a_{1}(0)=q
\end{gather*}
$$

Inserting Eq. (5) into Eq. (4) and equating the coefficients of the terms for the same $x$ powers, one arrives at the system of ordinary differential equations

$$
\begin{gather*}
\frac{d a_{0}}{d z}=\frac{i}{k} a_{2}-\frac{1}{2 k} a_{1}^{2}  \tag{6}\\
\frac{d a_{1}}{d z}=-\frac{2}{k} a_{1} a_{2}, \quad \frac{d a_{2}}{d z}=-\frac{k \eta}{p R}-\frac{2}{k} a_{2}^{2}
\end{gather*}
$$

This system has an analytic solution for any initial condition. It can be written as

$$
\begin{gather*}
a_{2}(z)=a_{2}(0) \frac{\tan \left(\alpha-z / z_{c}\right)}{\tan (\alpha)} \\
a_{1}(z)=a_{1}(0) \frac{\cos (\alpha)}{\cos \left(\alpha-z / z_{c}\right)} \\
a_{0}(z)=a_{0}(0)-\frac{i}{2} \ln \left(\frac{\cos (\alpha)}{\cos \left(\alpha-z / z_{c}\right)}\right)  \tag{7}\\
\\
-\frac{a_{1}^{2}(0) z_{c} \tan \left(z / z_{c}\right)}{2 k\left[1+\tan \alpha \tan \left(z / z_{c}\right)\right]} \\
\tan \alpha=
\end{gather*}
$$

The validity of this solution can be checked by direct substitution. Using initial conditions (5) and recurrence relations (7), one gets for the function $\tilde{P}(x, q, z)$
$\tilde{P}(x, q, z)=\exp \left(-i \frac{k t_{z}}{2 z_{c}} x^{2}\right) c_{z}^{-1 / 2} \exp \left(i \frac{x}{c_{z}} q-i \frac{z_{c} t_{z}}{2 k} q^{2}\right)$.
Hereinafter, the notation $s_{z}=\sin \left(z / z_{c}\right), c_{z}=\cos \left(z / z_{c}\right)$, and $t_{z}=\tan \left(z / z_{c}\right)$ is used.

Let us represent the general solution to the problem (for an arbitrary initial function) in the form of integral

$$
\begin{equation*}
A_{t}(x, y, z)=\int \frac{d q_{x} d q_{y}}{(2 \pi)^{2}} \tilde{P}_{t}\left(x, y, q_{x}, q_{y}, z\right) \tilde{A}_{0}\left(q_{x}, q_{y}\right) \tag{9}
\end{equation*}
$$

Substituting the expression for $\tilde{A}_{0}\left(q_{x}, q_{y}\right)$ in the form of the inverse Fourier transform and integrating with respect to $q_{x}$ and $q_{y}$, one obtains the desired integral transformation for a compound x-ray lens with the parabolic profile

$$
\begin{equation*}
A_{t}(x, y, z)=\int d x^{\prime} d y^{\prime} P_{t}\left(x, y, x^{\prime}, y^{\prime}, z\right) A_{t}\left(x^{\prime}, y^{\prime}, 0\right), \tag{10}
\end{equation*}
$$

whose propagator is factorized

$$
\begin{gather*}
P_{t}\left(x, y, x^{\prime}, y^{\prime}, z\right)  \tag{11}\\
=\exp (-i k \eta[d / p] z) P\left(x, x^{\prime}, z\right) P\left(y, y^{\prime}, z\right)
\end{gather*}
$$

and the partial propagator is determined by the expression

$$
\begin{gather*}
P\left(x, x^{\prime}, z\right)=\exp \left(-i \frac{\pi t_{z}}{\lambda z_{c}} x^{2}\right) \\
\times \frac{1}{\left(i \lambda z_{c} s_{z}\right)^{1 / 2}} \exp \left(i \pi \frac{\left(x-x^{\prime} c_{z}\right)^{2}}{\lambda z_{c} s_{z} c_{z}}\right) . \tag{12}
\end{gather*}
$$

Here, $\lambda=2 \pi / k$ is the x -ray wavelength.
This expression is the main result of the work. One can readily verify that this function transforms to the Dirac delta function $\delta\left(x-x^{\prime}\right)$ at $z=0$. Evidently, integral (10) must transform to the Kirchhoff integral in the limit $|\eta| \longrightarrow 0$. Indeed, after passing to the limit $\left|z_{c}\right| \longrightarrow \infty$ in Eq. (12), one obtains the following expression for the transverse part of the spherical wave in the paraxial approximation:

$$
\begin{align*}
& P\left(x, x^{\prime}, z\right)_{\mid \overrightarrow{|l|} \rightarrow 0} P_{K}\left(x-x^{\prime}, z\right) \\
& =\frac{1}{(i \lambda z)^{1 / 2}} \exp \left(i \pi \frac{\left(x-x^{\prime}\right)^{2}}{\lambda z}\right) . \tag{13}
\end{align*}
$$

The expression for the propagator in a more complex problem of radiation transfer in air at a distance of $r_{o}$ before the lens, through a lens of length $L$, and at a distance of $r_{i}$ in air after the lens can be written as a convolution

$$
\begin{gather*}
G\left(x, x^{\prime}, r_{o}, L, r_{i}\right) \\
=\int d x_{2} d x_{1} P_{K}\left(x-x_{2}, r_{i}\right) P\left(x_{2}, x_{1}, L\right) P_{K}\left(x_{1}-x^{\prime}, r_{o}\right) \tag{14}
\end{gather*}
$$

Note that the integrals in Eq. (14) are calculated analytically to give an analytic expression for the propagator $G$. However, the same result can be obtained by the method developed above, namely, by triply using recurrence relations (7), with the limiting transition $|\eta| \longrightarrow 0$
being used for air. The method of recurrence relations (7) is particularly suitable in the development of algorithms for computer simulation of imaging formation using a lens. Although this computer program was developed by us, the analysis of particular results is beyond the scope or this brief communication. Moreover, this method can also be applied to a system of lenses with different parameters.

Below, main features following from Eq. (12) for the operation of a compound refractive lens are considered. Taking into account that $\gamma=\beta / \delta \ll 1$, the complex parameter $z_{c}$ can be written as $z_{c}=\left(p F_{1}\right)^{1 / 2}(1+i \gamma / 2)$. At $L \ll\left(p F_{1}\right)^{1 / 2}$, one can retain only the leading terms in the sine and cosine expansions to obtain the following expression for the propagator in the thin-lens limit:

$$
\begin{gather*}
P_{0}\left(x, x^{\prime}, L\right)=\exp \left(-i \pi \frac{x^{2}}{\lambda F}[1-i \gamma]\right) P_{K}\left(x-x^{\prime}, L\right),  \tag{15}\\
F=\frac{F_{1}}{N}=\frac{R}{2 N \delta} .
\end{gather*}
$$

Note that the above-mentioned domain of applicability of this approximation can be written as $L \ll F$. Due to x-ray absorption, the plane wave, after passing through the lens, acquires the Gaussian shape, for which the intensity distribution halfwidth is $a_{\gamma}=0.664(\lambda F / \gamma)^{1 / 2}$. This value can be considered as the lens effective aperture. The expression including the terms on the order of $\left(L / z_{c}\right)^{3}$ can easily be written to obtain the corrections on the order of $L / F$ to the focal length in the thin-lens approximation.

For $L=L_{0}=\left(p F_{1}\right)^{1 / 2} \pi / 2$ and taking into account that $s_{z}=1$ and $c_{z}=i \gamma \pi / 4$, one obtains, in the linear approximation in $\gamma$, the following expression for the propagator:

$$
\begin{gather*}
P\left(x, x^{\prime}\right)=\frac{1}{\left(i \lambda z_{c}\right)^{1 / 2}} \\
\times \exp \left[-i \frac{2 \pi}{\lambda z_{c}} x x^{\prime}-\gamma \frac{\pi^{2}}{4 \lambda z_{c}}\left(x^{2}+x^{\prime 2}\right)\right] \tag{16}
\end{gather*}
$$

From this expression, it follows that, when passing through this lens, the wave is modulated by a Gaussian function because of the absorption in lens and then turns to its Fourier transform. In particular, at the lens output, a plane wave has a Gaussian intensity distribution with the halfwidth $s_{\gamma}=0.47\left(\lambda L_{0} \gamma\right)^{1 / 2}$, and the focal length is $L_{0}$. The quantity $s_{\gamma}$ gives the focal spot diameter, whereas the lens effective aperture in this case is $a_{\gamma}=0.846\left(\lambda L_{0} / \gamma\right)^{1 / 2}$. If the absorption is ignored, the propagator will be equal to delta function $\delta\left(x+x^{\prime}\right)$ for $L=2 L_{0} ;$ i.e., the wave field is restored in the inverted form. Clearly, the lens will have the same properties for
$L=3 L_{0}, 4 L_{0}$, and so on. However, the absorption diminishes the lens working region with increasing $L$.

In conclusion, let us estimate the parameters of the system. Consider a compound aluminum lens for photon energies of 25 keV . Let $p=1 \mathrm{~mm}$ and $R=0.2 \mathrm{~mm}$ [2]. One has in this case $\gamma=2.02 \times 10^{-3}, F_{1}=116 \mathrm{~m}$, and $L_{0}=53.4 \mathrm{~cm}$. Therefore, the critical size of the compound lens is achieved when 534 elements are used. Evidently, $L_{0}$ is the minimal attainable focal length for a given radius of curvature of the parabolic surfaces. The focal spot diameter in this case is $s_{\gamma}=0.11 \mu \mathrm{~m}$, and the effective aperture is $a_{\gamma}=97 \mu \mathrm{~m}$. Smaller focal length can be obtained by a gradual decrease in the radius of curvature of the surfaces in individual elements.

REFERENCES

1. A. Snigirev, V. Kohn, I. Snigireva, and B. Lengeler, Nature 384, 49 (1996).
2. B. Lengeler, C. G. Schroer, M. Richwin, et al., Appl. Phys. Lett. 74, 3924 (1999).
3. Y. Ohishi, A. Q. R. Baron, M. Ishii, et al., Nucl. Instrum. Methods Phys. Res. A 467-468, 962 (2001).
4. V. V. Protopopov and K. A. Valiev, Opt. Commun. 151, 297 (1998).
5. V. Kohn, I. Snigireva, and A. Snigirev, Phys. Rev. Lett. 85, 2745 (2000).

Translated by V. Sakun


[^0]:    (left) Compound refractive x-ray lens and (right) parameters of its individual elements.

